Chaotic and Periodic Motions of Satellites in Elliptic Orbits

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The spinning motion of aspherical satellites whose center of mass moves in a given elliptic polar orbit around an oblate central body is investigated using analytical and numerical methods. In the case of a magnetic satellite, dipole-dipole interaction with the central body is included. For small eccentricity, oblateness and magnetic interaction, the Melnikov method is used to study chaotic and perodic motions. The parameter dependence of the width of the chaotic layer and of the periodic resonances is discussed. For some selected parameter values the theoretical predictions are checked by numerical methods.

Key words: Satellite motion, subharmonics, homoclinic points, Melnikov method, chaos

1. Introduction

It is well known that the motion of celestial bodies under the influence of gravitational forces may show not only periodic but also chaotic behaviour. Already Poincaré, while studying the three body problem [1], found a number of interesting and complicated motions. His discovery of homoclinic motions near periodic solutions marks the beginning of the modern theory of nonlinear dynamics. Also in the solar system there are several physical situations where chaotic behaviour plays an important role. Recently a series of papers were published giving numerical evidence and experimental indications of such a behaviour. The examples treated are: orbits of meteorites from the asteroid belt [2–5], the long term motion of Pluto [6, 7] and comet Halley [8], the chaotic tumbling of Hyperion [2, 9–11], one of Saturn's more distant satellites, and other aspherical satellites [2, 12].

The last example can be described in a simplified plane model. In this model the orbits are taken to be fixed ellipses and it is assumed that the timescale for variations in the spin rate is much shorter than for variations in satellite orbits. Moreover, it is assumed that the spin axis coincides with one of the main inertia axes and is perpendicular to the orbital plane. This results in a quite simple equation of motion, a perturbed pendulum equation.

In this paper we use analytical and numerical methods to study the plane motions of aspherical satellites in elliptic orbits of small eccentricity and under the influence of further perturbations. For these cases the Melnikov method [13, 14] gives the possibility to estimate the width of periodic resonances and of chaotic layers. The method is applicable to small time periodic perturbations of integrable systems which have periodic and/or heteroclinic (homoclinic) solutions. A heteroclinic solution connects two different fixed points. It can be used to calculate the corresponding heteroclinic Melnikov function of the perturbed system. Simple zeros of this function indicate transverse intersections of stable and unstable manifolds. According to the Smale-Birkhoff theorem [14], which is also valid for homoclinic cycles formed of transverse heteroclinic orbits, chaotic solutions exist. The amplitude of the oscillating part of the Melnikov function is a measure of the maximum separation of the manifolds and characterizes therefore the width of the chaotic layer.

With the periodic solutions of the unperturbed system which are in resonance with the external perturbation one calculates the subharmonic Melnikov functions, whose simple zeros give notice of the existence of subharmonic orbits. The width of the subharmonic resonances depends on the amplitude of the corresponding Melnikov functions. It should be noted that the same results can be obtained with first order averaging.

In Sect. 2 the equations of motion of a satellite which moves in an elliptical orbit around an oblate central body is investigated. Magnetic dipole-dipole interaction which influences the motion of artificial satellites is included in the model discussed in Section 3. In both cases plane motion is permitted by the basic Euler equations only for polar satellites. The analytical predictions of the Melnikov method are discussed and checked by some numerical experiments in Section 4.

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2. Satellite in an Elliptic Orbit Around an Oblate Central Body

The equations of motion of an aspherical satellite whose centre of mass moves in an elliptic orbit around an oblate axially symmetric central body are given by Euler's equations with external torque which is produced by gravity [15–17]. In the case of a polar satellite, i.e. a satellite whose orbital plane contains the symmetry axis of the central body, a spinning motion around the axis of the principal moment of inertia which is perpendicular to the orbital plane is possible. Then the set of Euler's equations reduces to a single equation:

$$(1+e\cdot\cos(\tau))\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}+\frac{3b}{2}\sin(2\theta)=2e\cdot\left(1+\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\sin(\tau)+\frac{\beta\, b(1+e\cos(\tau))^2}{4(1-e^2)^2}$$
$$\cdot\{16\cos(2\theta)\sin 2(\tau+\tau_\pi)+\sin(2\theta)\,[5-19\cos 2(\tau+\tau_\pi)]\} \tag{2.1}$$

Here $\theta(\tau)$ specifies the orientation of the satellite. The independent variable τ is the true anomaly, e the eccentricity of the ellipse and $b=(A-C)/B$ includes the three principal moments of inertia A, B, C ($|b|\leq 1$). B is the moment which is related to the axis perpendicular to the orbital plane. One needs only to consider $b\geq 0$ because (2.1) is invariant with respect to $\theta\to\theta+\pi/2$, $b\to -b$. τ_π is the angle between the perihelion and the ascending node, and the parameter β designates the deviation of the central body from the spherical symmetry. $\beta=(R_\mathrm{E}-R_\mathrm{N})/R_\mathrm{N}$, where R_N is the polar radius and R_E the equatorial radius of the central body. In (2.1) only the first two terms in the Legendre series expansion of the gravitational potential are considered. Figure 1 shows the coordinates of the satellite.

The prerequisite to the application of the Melnikov method is that the equations of motion can be written as the sum of an autonomous system with heteroclinic (homoclinic) orbits and a small time periodic perturbation. To find such a form, we introduce a small parameter $\varepsilon(0\leq\varepsilon\ll 1)$ by the substitution $e=\varepsilon\, e'$ and $\beta=\varepsilon\,\beta'$.

A power series expansion of (2.1) yields up to the first order in the small parameter ε

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}+\frac{3b}{2}\sin(2\theta)=\varepsilon\{2e'\left(1+\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)\sin(\tau)+\frac{3b\,e'}{2}\cos(\tau)\sin(2\theta)$$
$$+\frac{b\,\beta'}{4}\sin(2\theta)\,[5-19\cos 2(\tau+\tau_\pi)]+4b\,\beta'\cos(2\theta)\sin 2(\tau+\tau_\pi)\}+0(\varepsilon^2). \tag{2.2}$$

In this form the equation of motion is valid for small eccentricity and oblateness only. Moreover, it is assumed that the ratio of the principal moments of inertia b is not a small quantity. The corresponding unperturbed system ($\varepsilon=0$) is integrable and has heteroclinic orbits connecting the hyperbolic saddle points at $(\theta,\dot\theta)=(\pi/2+n\,\pi,0)$ and $(\theta,\dot\theta)=(-\pi/2+n\,\pi,0)$, $n=0,\ \pm 1,\ \pm 2,\ldots$. These orbits are embedded in a continuous sequence of periodic orbits. With regard to their energy, the unperturbed solutions can be divided into three types: the oscillatory, heteroclinic and rotary motion type.

The Poincaré map of the perturbed system (2.2), for ε sufficiently small, has unique hyperbolic saddle points near the unperturbed ones [14].

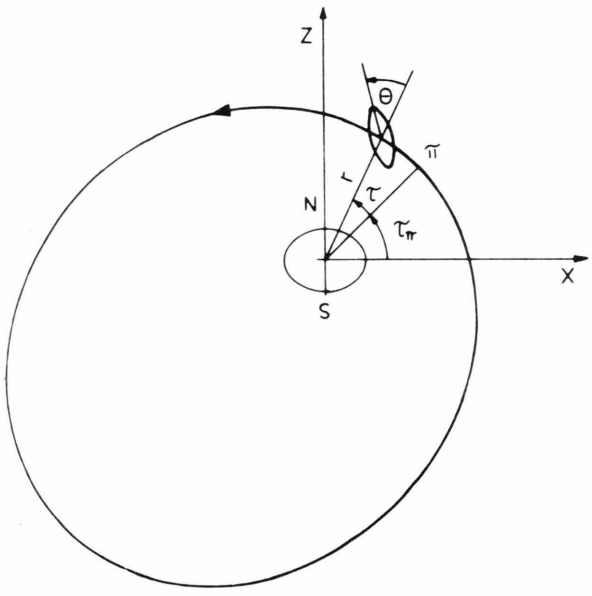

Fig. 1. Position of the orbital plane relative to the geocentric coordinate system (X, Y, Z).

For the heteroclinic Melnikov function, which is a measure of the distance (first order of the perturbation theory) between the stable and unstable manifolds of hyperbolic fixed points in the Poincaré map, we obtain

$$M^{\pm}(\tau_0) = \int_{-\infty}^{\infty} \frac{d\theta}{d\tau} \left\{ 2e' \left(1 + \frac{d\theta}{d\tau}\right) \sin(\tau) + \frac{3be'}{2} \cos(\tau) \sin(2\theta) + \frac{b\beta'}{4} \sin(2\theta) \left[5 - 19\cos 2(\tau + \tau_\pi)\right] + 4b\beta' \cos(2\theta) \sin 2(\tau + \tau_\pi) \right\} d\tau .$$

After inserting the unperturbed separatrix solution $\theta(\tau - \tau_0)$ and integrating we have

$$M^{\pm}(\tau_0) = \frac{\pi e'}{2} \sin(\tau_0) \left[3\,\mathrm{csch}\left(\frac{\pi}{2\sqrt{3b}}\right) \pm 4\,\mathrm{sech}\left(\frac{\pi}{2\sqrt{3b}}\right) \right] + \frac{\pi\beta'}{3} \sin 2(\tau_0 + \tau_\pi) \left[19\,\mathrm{csch}\left(\frac{\pi}{\sqrt{3b}}\right) \pm 16\,\mathrm{sech}\left(\frac{\pi}{\sqrt{3b}}\right) \right], \tag{2.3}$$

where the sign $\pm$ stands for the upper and lower separatrix, respectively. Obviously both functions have simple zeros with respect to τ_0. Therefore one finds transverse intersections of the stable and unstable manifolds, and hence according to the Smale-Birkhoff theorem sets of chaotic orbits in the neighbourhood of the unperturbed separatrix for all sufficiently small ε. The special case without oblateness $\beta' = 0$ can be found in [18], but it is already presented in [19].

The subharmonic Melnikov functions can be obtained in the same way using the various periodic solutions of the unperturbed problem. The external period of the perturbation is 2π. Therefore the resonance conditions for the oscillatory motion are

$$4K(k)/\sqrt{3b} = 2\pi m/n \tag{2.4}$$

(m, n relatively prime natural numbers) and

$$2K(1/k)/k\sqrt{3b} = 2\pi m/n \tag{2.5}$$

for the rotary solutions. $K(k)$ denotes the complete elliptic integral of the first kind, k is the elliptic modulus which parametrizes the unperturbed periodic solutions. The integrals can be computed making use of the method of residues, and for the oscillatory motion one obtains

$n = 1$

$$M^m_{\mathrm{osc}}(\tau_0) = 4\pi e' \sin(\tau_0)\,\mathrm{sech}(K'(k)/\sqrt{3b}), \quad m = 2, 4, \ldots,$$

$$M^m_{\mathrm{osc}}(\tau_0) = 3\pi e' \sin(\tau_0)\,\mathrm{csch}(K'(k)/\sqrt{3b}), \quad m = 1, 3, 5, \ldots.$$

$n = 2$

$$M^{m/2}_{\mathrm{osc}}(\tau_0) = 4\pi\beta'(19/3) \sin 2(\tau_0 + \tau_\pi)\,\mathrm{csch}(2K'(k)/\sqrt{3b}), \quad m = 2, 4, \ldots,$$

$$M^{m/2}_{\mathrm{osc}}(\tau_0) = 4\pi\beta'(16/3) \sin 2(\tau_0 + \tau_\pi)\,\mathrm{sech}(2K'(k)/\sqrt{3b}), \quad m = 1, 3, 5, \ldots, \tag{2.6}$$

where $K'(k) = K(k')$ and $(k')^2 = 1 - k^2$.

From the resonance conditions (2.4) and the inequality $K(k) \geq \pi/2$ follows that there is a minimum value of the parameter b for which the resonance conditions can be fulfilled. This minimum value is $b = 1/3\,m^2$ for $n = 1$ and $b = 4/3\,m^2$ for $n = 2$. Such a condition does not exist for the subharmonics in the rotary regime. In this case the Melnikov functions are given by

$n = 1$

$$M^m_{\mathrm{rot}}(\tau_0) = \frac{\pi e'}{2} \sin(\tau_0) \left[\pm 4\,\mathrm{sech}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right) + 3\,\mathrm{csch}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right) \right] \qquad m = 1, 2, 3, \ldots.$$

$n = 2$

$$M^{m/2}_{\mathrm{rot}}(\tau_0) = \frac{2\pi\beta'}{3} \sin 2(\tau_0 + \tau_\pi) \left[19\,\mathrm{csch}\left(\frac{2K'(1/k)}{k\sqrt{3b}}\right) \pm 16\,\mathrm{sech}\left(\frac{2K'(1/k)}{k\sqrt{3b}}\right) \right] \qquad m = 1, 2, 3, \ldots. \tag{2.7}$$

In the limit $m \to \infty$ the heteroclinic Melnikov function (2.3) can be reproduced by

$$\lim_{m\to\infty} \sum_{n=1}^{2} \frac{1}{n} M_{\mathrm{rot}}^{m/n}(\tau_0) = M^{\pm}(\tau_0). \tag{2.8}$$

This limit is more complicated for the oscillatory Melnikov functions (2.6) because a distinction between the even and odd subharmonics is necessary.

The knowledge of the Melnikov functions is important because it gives direct information on the width of the periodic resonances and that of the chaotic layers. Up to the first order in the small parameter ε the maximum separation of the upper (lower) branches of the corresponding stable and unstable manifolds is given by [14]

$$\delta^{\pm} = \varepsilon \cdot \frac{\max_{\tau_0 \in R} |M^{\pm}(\tau_0)|}{|f(q_0(0))|}, \tag{2.9}$$

where the denominator is the norm of the unperturbed vector field ($\dot{x} = f(x)$) calculated with the separatrix solution q_0 at the point $\tau = \tau_0$. The larger value of the two separations δ^+ and δ^- determines the width of the chaotic layer.

For the analysis of global stochasticity with the Chirikov resonance overlap criterion the width of the periodic resonances has to be calculated.

On the basis of first order averaging one obtains for the width ΔI, measured in the action variable I (see [20]),

$$\Delta I = 2\sqrt{\varepsilon \cdot 2(V_{\max} - V_{\min})/\Omega'}. \tag{2.10}$$

Ω' is the derivative of the frequency of the unperturbed system with respect to the action variable taken at the resonance value $I^{m/n}$, i.e.

$$\Omega' = \left.\frac{\mathrm{d}\Omega}{\mathrm{d}I}\right|_{I^{m/n}}.$$

$V_{\max}$ and $V_{\min}$ are the maxima and minima of

$$V(\Phi) = \frac{1}{2\pi n} \int M^{m/n}(\Phi/\Omega^{m/n})\, \mathrm{d}\Phi.$$

Here $M^{m/n}$ is the Melnikov function of the corresponding subharmonic resonance which is characterized by the resonance frequency $\Omega^{m/n} = n\,\omega/m$ (ω = frequency of external excitation). Usually the bandwidths are observed in the $(\theta, \dot{\theta})$ Poincaré section ($\dot{\theta} = \mathrm{d}\theta/\mathrm{d}\tau$). Therefore it is necessary to convert the width (2.10) in action space back to $(\theta, \dot{\theta})$ space. This can be done by [20]

$$\Delta\dot{\theta} = \left.\frac{\partial\dot{\theta}}{\partial H_0}\right|_{\theta=\theta_{\max}} \cdot \left.\frac{\partial H_0}{\partial I}\right|_{I=I^{m/n}} \cdot \Delta I, \tag{2.11}$$

where H_0 is the Hamiltonian of the corresponding unperturbed system (see (2.2)), i.e.

$$H_0 = \frac{\dot{\theta}^2}{2} - \frac{3b}{4}\cos(2\theta),$$

which can be written as a function of the action variable I. $\theta_{\max}$ is the value of θ where the maximum bandwidth occurs. Because for large m and $\omega = 1$ (see (2.1)) we have [19, 21]

$$\Omega' \sim \frac{\exp(m\pi\sqrt{3b}/n)}{(\sqrt{3b}\,m/n)^3},$$

the bandwidths decrease rapidly as m increases.

Using the Melnikov functions (2.6) or (2.7), the width of the resonances can be calculated by (2.10) and (2.11). Here we give a simple example for the oscillatory regime with $n = 1$, $m = 2, 4, \ldots$. For this case the following expressions are obtained

$$M_{osc}^{m}(\tau_0) = 4\pi e' \sin(\tau_0)\,\mathrm{sech}\left(\frac{K'(k)}{\sqrt{3b}}\right),$$

$$V_{\max} - V_{\min} = \frac{4e'}{m}\,\mathrm{sech}\left(\frac{K'(k)}{\sqrt{3b}}\right),$$

$$\Omega' = \frac{-\pi^2}{4k^2k'^2K^3(k)}[E(k) - k'^2K(k)] \quad \text{(see [19])}.$$

$E(k)$ denotes the complete elliptic integral of the second kind. Insertion of these results in (2.10) gives

$$\Delta I = 8 \cdot \frac{kk'K(k)}{\pi}\sqrt{\varepsilon \cdot \frac{2K(k)\,e'}{m[E(k) - k'^2K(k)]}\,\mathrm{sech}\left(\frac{K'(k)}{\sqrt{3b}}\right)}.$$

According to (2.11) the maximum bandwidth is given in units of $\dot{\theta}$ at $\theta_{\max} = 0$ (s. Fig. 7),

$$\Delta\dot{\theta} = \frac{1}{k\sqrt{3b}} \cdot \frac{\pi\sqrt{3b}}{2K(k)}\Delta I$$

$$= 4k'\sqrt{\frac{\varepsilon \cdot 2K(k)\,e'}{m[E(k) - k'^2K(k)]}\,\mathrm{sech}\left(\frac{K'(k)}{\sqrt{3b}}\right)}.$$

Using the same method, Veerman and Holmes [20] calculated the width of the resonance bands for a pair of linearly coupled pendula and compared their predictions with numerical computations. Unfortunately,

their formulas are useful only for very small coupling ($\varepsilon \leq 10^{-4}$). We have not performed such numerical tests, but we expect similar results and consider our results on resonant bandwidths for larger ε as a rough estimation only.

3. Polar Satellites with Magnetic Dipole-dipole Interaction

Artificial satellites are influenced not only by gravity but also by magnetic and aerodynamic forces. As a rule, the effect of radiation pressure and micrometeorites can be neglected. For altitudes of more than 500 km gravitational and magnetic forces dominate [22]. The most important magnetic interactions are the interactions of a given constant and an induced dipole moment of the satellite with the magnetic field of the earth. In the following we discuss the plane motion of a polar satellite in an elliptic orbit oriented as in Sect. 2 and additionally perturbed by the interaction between a constant magnetic dipole of the satellite and the magnetic dipole of the earth. The satellite dipole coincides with an axis of the moment of inertia in the orbital plane. The magnetic field of the earth is described by a dipole which is situated in the mass centre. In this simplified problem the equation of motion for the orientation in the plane of motion reads [22]

$$(1 + e \cdot \cos(\tau)) \frac{d^2\theta}{d\tau^2} + \frac{3b}{2} \sin(2\theta) = 2e \cdot \sin(\tau)\left(1 + \frac{d\theta}{d\tau}\right) + \frac{\alpha}{2}[3\cos(\theta - \tau + \tau_\pi) - \cos(\theta + \tau + \tau_\pi)]. \tag{3.1}$$

In (3.1) we have neglected the influence of oblateness of the central body. α is the parameter of magnetic coupling between satellite and earth. Because of the invariance of (3.1) with respect to the transformation $\alpha \to -\alpha$, $\theta \to \theta + \pi$ we need to consider the case $\alpha > 0$ only. $b > 0$ ($b < 0$) means that the dipole coincides with the axis of the smaller (larger) moment of inertia. To apply the Melnikov method, it is necessary to assume that the eccentricity e and the dipole parameter α are small quantities, i.e. $e = \varepsilon e'$ and $\alpha = \varepsilon \alpha'$ with e', $\alpha' = 0(1)$. Many artificial satellites are characterized by small α, e.g. $\alpha \approx 0.14$ applies to the third Soviet sputnik. There are also many examples where α is not as small as that [22].

As in Sect. 2 the heteroclinic Melnikov function is calculated using the heteroclinic solutions of the unperturbed system. For $b > 0$ one obtains

$$M^{\pm}(\tau_0) = \frac{1}{2}\pi e' \sin\tau_0 \left[3\,\mathrm{csch}\left(\frac{\pi}{2\sqrt{3b}}\right) \pm 4\,\mathrm{sech}\left(\frac{\pi}{2\sqrt{3b}}\right)\right] + \frac{\alpha'\pi}{\sqrt{3b}} \cos(\tau_0 + \tau_\pi)\left[\pm\,\mathrm{csch}\left(\frac{\pi}{2\sqrt{3b}}\right) + 2\,\mathrm{sech}\left(\frac{\pi}{2\sqrt{3b}}\right)\right]. \tag{3.2}$$

Here, as in (2.3), the positive (negative) sign characterizes the upper (lower) branches of the manifolds. In the case of $b < 0$ one has to replace $\cos(\tau_0 + \tau_\pi)$ by $\sin(\tau_0 + \tau_\pi)$ and b by $-b$. Also the subharmonic Melnikov functions can be calculated. One obtains for the even and odd subharmonics, respectively, in the oscillatory regime

$$M^m_{\mathrm{osc}}(\tau_0) = 4\pi\left[e' \sin\tau_0 + \frac{\alpha'}{\sqrt{3b}}\cos(\tau_0 + \tau_\pi)\right]\mathrm{sech}\left(\frac{K'(k)}{\sqrt{3b}}\right). \qquad m = 2, 4, 6, \ldots,$$

$$M^m_{\mathrm{osc}}(\tau_0) = 3\pi\left[e' \sin\tau_0 + \frac{2\alpha'}{3\sqrt{3b}}\cos(\tau_0 + \tau_\pi)\right]\mathrm{csch}\left(\frac{K'(k)}{\sqrt{3b}}\right). \qquad m = 1, 3, 5, \ldots, \tag{3.3}$$

and in the rotary regime

***n* = 1**

$$M^m_{\mathrm{rot}}(\tau_0) = \frac{1}{2}\pi e' \sin\tau_0 \left[\pm 4\,\mathrm{sech}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right) + 3\,\mathrm{csch}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right)\right] \qquad m = 1, 2, 3, \ldots,$$

***n* = 2**

$$M^{m/2}_{\mathrm{rot}}(\tau_0) = \frac{2\pi\alpha'}{k^2\sqrt{3b}}\cos(\tau_0 + \tau_\pi)\left[\pm\,\mathrm{csch}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right) + 2\,\mathrm{sech}\left(\frac{K'(1/k)}{k\sqrt{3b}}\right)\right] \qquad m = 1, 3, 5, \ldots. \tag{3.4}$$

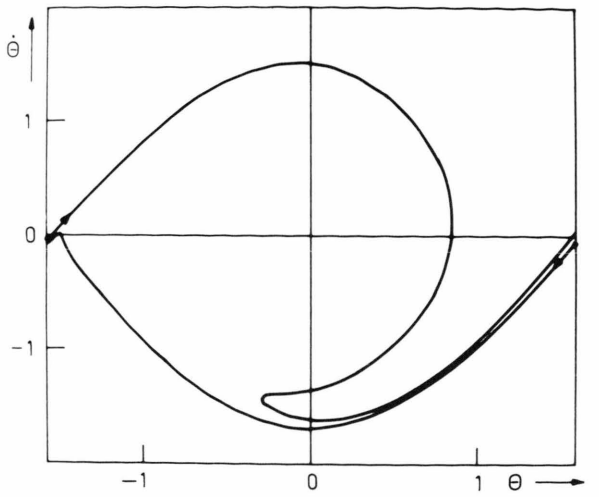

Fig. 2. Parts of the unstable manifolds of the hyperbolic fixed points corresponding to the Poincaré map of (2.2); parameter set: $\varepsilon = 0.1$, $e' = 1$, $\beta' = 0$, $b = 1$.

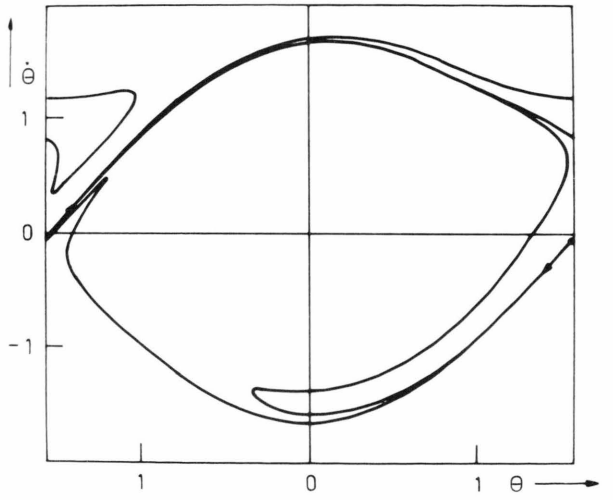

Fig. 3. Parts of the unstable manifolds of the hyperbolic fixed points corresponding to the Poincaré map of (2.2); parameter set: $\varepsilon = 0.04$, $e' = 0$, $\beta' = 1$, $b = 1$, $\tau_\pi = \pi/2$.

The k-values are calculated by means of the resonance conditions (2.4) and (2.5). The positive (negative) sign is valid for rotations in the positive (negative) direction. In the limit $m \to \infty$ the heteroclinic Melnikov function (3.2) can be obtained with the aid of (2.8).

4. Discussion and Numerical Experiments

In order to test the validity of the analytical predictions, some comparisons with numerical calculations

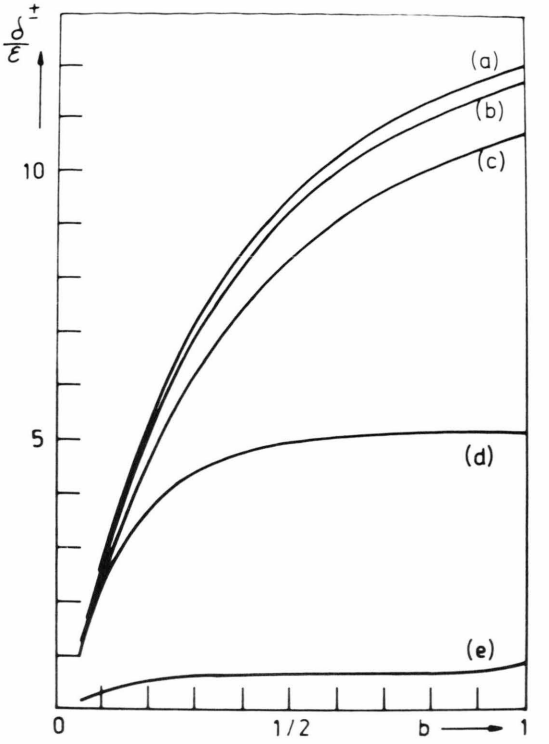

Fig. 4. Dependence of $\delta^\pm$ on the parameter b according to (2.2).
(a) δ^+: $\tau_\pi = \pi/4$; $e' = \beta' = 1$, (b) δ^+: $\tau_\pi = \pi/8, 3\pi/8$; $e' = \beta' = 1$, (c) δ^+: $\tau_\pi = 0, \pi/2$; $e' = \beta' = 1$, (d) δ^+: $e' = 1$, $\beta' = 0$, (e) δ^-: $\tau_\pi = 0, \pi/2$; $e' = \beta' = 1$.

have been carried out. Figures 2 and 3 show parts of the unstable manifolds of the hyperbolic fixed points corresponding to the Poincaré map of (2.2) in the limit cases $\beta' = 0$ and $e' = 0$, respectively. The stable manifolds can be obtained by reflecton at the axis $\theta = 0$ because the equation of motion (2.2) is invariant with respect to the transformation $(\theta, \tau) \to (-\theta, -\tau)$ in the case $\tau_\pi = \pi/2$. Therefore the predicted crossing of the unstable and stable manifolds can be observed.

The dependence of the width of the chaotic layer upon the parameters b and τ_π is shown in Fig. 4 for the example of Section 2. The small value of δ^- indicates that the width of the chaotic layer is determined by δ^+. The Melnikov function (2.3) has the property $M^\pm(\tau_0, \tau_\pi) = -M^\pm(\tau_0 + \pi, \tau_\pi + \pi/2)$, and therefore only the interval $0 \le \tau_\pi < \pi/2$ is of interest. A more complicated situation (see Figs. 5 and 6) is possible for the satellites with magnetic interaction. For instance for $b \approx 0.2425$, $\tau_\pi = \pi/2$ and $\alpha' = 1$ the width δ^+ vanishes.

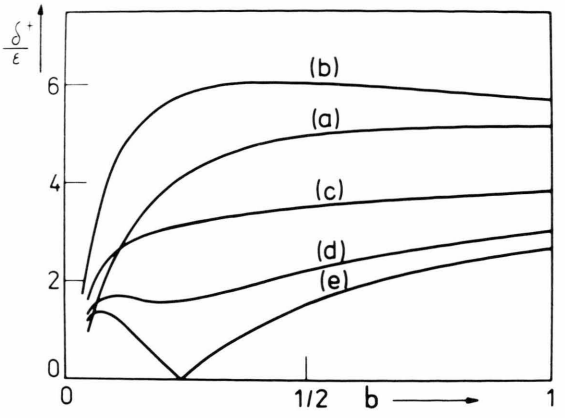

Fig. 5. Dependence of the width of the chaotic layer δ^+ on the parameter b according to (3.1).
(a) $\alpha' = 0$, (b) $\tau_\pi = 0$, $\alpha' = 1$, (c) $\tau_\pi = \pi/4$; $\alpha' = 1$, (d) $\tau_\pi = 3\pi/8$, $\alpha' = 1$, (e) $\tau_\pi = \pi/2$, $\alpha' = 1$.

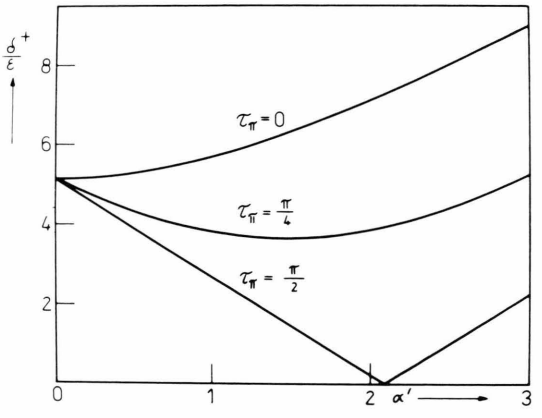

Fig. 6. Dependence of the width of the chaotic layer δ^+ upon the parameter α' for a fixed value $b = 1$ according to (3.1).

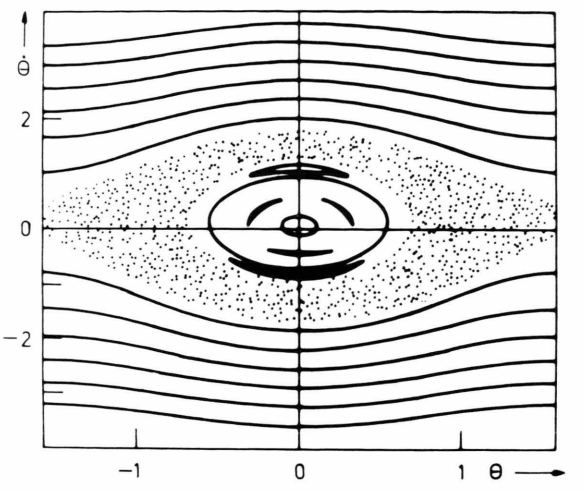

Fig. 7. Poincaré section of (2.2) with $\varepsilon = 0.1$, $e' = 1$, $\beta' = 0$, $b = 1$.

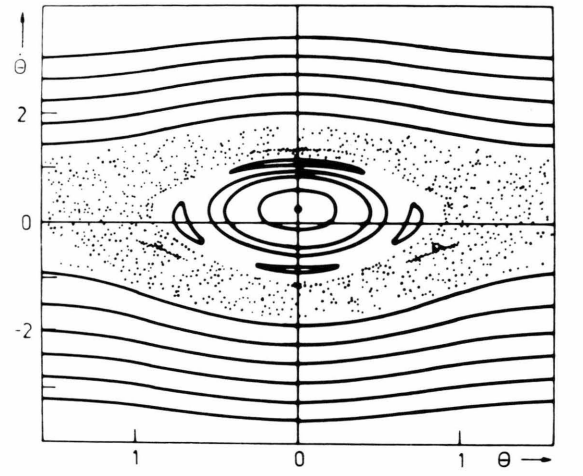

Fig. 8. Poincaré section of (2.2) with $\varepsilon = 0.04$, $e' = 0$, $\beta' = 1$, $b = 1$, and $\tau_\pi = \pi/2$.

Figures 7 and 8 show the Poincaré section for selected initial values of (2.2). Clearly visible are the chaotic region around the unperturbed separatrix solution, some of the KAM-tori and resonances. The width of the chaotic layers is in agreement with (2.9). On the other hand, the width of the resonances is smaller than the width of the chaotic layers.

The subharmonic Melnikov functions of Sects. 2 and 3 indicate the different nature of the perturbation terms in the equations of motion. For instance in (2.2) the parametric terms contribute to the even subharmonics only and the additive term $2e'\sin(\tau)$ to the odd subharmonics in the case $n = 1$. The influence of the oblateness is given by the $n = 2$ contributions, which arises from the physical symmetry of the problem. A similar situation occurs also for the satellites with magnetic interactions.

The existence of subharmonic solutions of (2.2) has already been shown in [23] for the special case $\beta' = 0$ by using an averaging method. A comparison with our equations (2.6) is difficult because the final result in [23] has a very detailed structure originating from a power series expansion of the corresponding integrals. It must be underlined that within the resonances there

are further regions in which separatrix crossing occurs. However, this crossing cannot be detected with the Melnikov method because exponentially small Melnikov functions appear [14].

The model of the plane motions of satellites contains two simplifications which restrict the validity of our results. As a first point, the spin-orbit coupling cannot generally be neglected. For instance, the oblateness of the central body causes a shift of the perihelion and a rotation of the orbital plane. For many natural and artificial satellites these deviations from Keplerian motion are very small; especially for polar satellites the rotation of the orbital plane vanishes [24]. Moreover, we expect that the plane problem with the fixed spin axis is, in general, an unstable one so that the full set of the Euler equations must be used. An indication of this instability can be found in the numerical experiments of Wisdom et al. [9].

Up to now we have not considered any dissipative effects. Actually, satellites are exposed to several frictional forces. Tidal dissipation in natural satellites produces a torque T which may be represented by

$$T = -\frac{c}{r^6} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t} \qquad (c = \text{const}).$$

r is the instantaneous radius (see Figure 1).
This so called Mac Donald torque results in an additional term in (2.1) which is proportional to $\mathrm{d}\theta/\mathrm{d}\tau$. If the damping constant is small but of the same order as eccentricity and oblateness, one obtains a τ_0 independent item in the Melnikov functions. From this we can calculate bifurcation conditions for the appearance of heteroclinic and subharmonic solutions in the same manner as for other periodically excited oscillator systems with damping [14]. The corresponding calculations are simple and not shown here. The same is true if we take a more complicated velocity dependence in the damping term which for instance is caused by the aerodynamic damping of artificial satellites.

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